

# Kolmogorov Complexity and Diophantine Approximation

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# Diophantine Approximation

**Diophantine Approximation** classifies real numbers by how well they may be approximated by rational numbers.

Measured in terms of denominator:

$$\left| x - \frac{p}{q} \right| < F(q)$$

For which  $F$  does this have **infinitely many solutions**  $p/q$  (with  $p, q$  relatively prime)?

Most commonly:  $F(q) = 1/q^\delta$

# Dirichlet

For any irrational number  $\alpha$  there exist infinitely many rational numbers  $p/q$  such that

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2}$$

Such a sequence is given by the convergents of the continued fraction expansion

$$\alpha = [a_0; a_1, \dots] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}} \quad a_i \in \mathbb{Z}^+$$

# Measuring Irrationality

The **irrationality exponent** of a real number  $x$  is defined as

$$\delta(x) = \sup \left\{ \delta : \exists^\infty p, q \left| x - \frac{p}{q} \right| < \frac{1}{q^\delta} \right\}.$$

Every irrational number has irrationality exponent  $\geq 2$   
[Dirichlet].

A **Liouville number** is defined by the property  $\delta(x) = \infty$ .

Other examples:

–  $\delta(e) = 2$

–  $\delta(\pi) \leq 7.60630853$

# The Metric Theory

**Khinchin:** Almost every real has irrationality exponent 2.

In fact, let  $\Psi : \mathbb{R}^{\geq 1} \rightarrow \mathbb{R}^{>0}$  be a continuous function such that  $x \mapsto x^2\Psi(x)$  is non-increasing. Then the set

$$\Delta(\Psi) = \left\{ x : \exists^\infty p, q \left| x - \frac{p}{q} \right| < \Psi(q) \right\}$$

has full Lebesgue measure if

$$\sum_q q\Psi(q) = \infty,$$

and has Lebesgue measure zero otherwise.

**Jarník**, and independently **Besicovitch**, showed

$$\dim_{\mathbb{H}}\{x: \delta(x) \geq \delta\} = \frac{2}{\delta}$$

Jarník's proof actually shows that

$$\dim_{\mathbb{H}}\{x: \delta(x) = \delta\} = \frac{2}{\delta}.$$

Let

$$G_q(a) = \left\{ x \in \left( \frac{1}{q^a}, 1 - \frac{1}{q^a} \right) : \exists p \left| x - \frac{p}{q} \right| \leq \frac{1}{q^a} \right\}.$$

For  $n$  sufficiently large,  $q_1 \neq q_2$  prime and  $n < q_1, q_2 \leq 2n$ ,

$$G_{q_1}(a) \cap G_{q_2}(a) = \emptyset \text{ with gaps } \geq \frac{1}{8n^2}$$

If we let

$$H_n(a) = \bigcup_{\substack{q \text{ prime} \\ n < q \leq 2n}} G_q(a),$$

and let  $(n_i)$  be a sequence of natural numbers, then

$$\bigcap H_{n_i}(a)$$

is a **Cantor set** (after some trimming) containing only reals with irrationality exponent  $\geq a$ .

One can show that if  $(n_i)$  is sufficiently fast growing, this Cantor set has dimension at least  $2/a$ .



# Mass Distribution Principle

Let  $s > 0, X \subseteq \mathbb{R}$ . If  $\mu$  is a probability measure on  $\mathbb{R}$  such that  $0 < \mu(X) < \infty$ , and there exist  $\varepsilon, c > 0$  such that for every interval  $I$  with  $|I| < \varepsilon$ ,

$$\mu(I) \leq c\varepsilon^s,$$

then

$$\dim_H(X) \geq s.$$

## Mass Distribution on Jarník's Fractal

We can uniformly distribute a (unit) mass along a Cantor set and get a bound for the measure of  $|I|$  from

- the number of subintervals in each step
  - for Jarník's fractal: prime number theorem,  $\geq n/(2 \log n)$ ,
- the length of gaps between intervals.

# Effective Dimension

The irrationality exponent reflects how well a real can be approximated by rational numbers.

**Information theoretically:** Think of  $(p, q)$  as a description of a real with respect to a very simple decoder:  $(p, q) \mapsto p/q$ .

The **effective dimension** [Lutz] reflects how well a real can be approximated by arbitrary effective decoders:

$$\dim_{\text{H}}(x) = \liminf_{n \rightarrow \infty} \frac{K(x \upharpoonright n)}{n}$$

## Effective Dimension and Irrationality Exponent

For a random real  $x$ ,  $p/q$  cannot give significantly more than  $2 \log q$  bits of information about  $x$ .

Hence almost every real has irrationality exponent 2.

If  $x \in (0, 1)$  is Liouville, on the other hand, for every  $n$  there exist  $p/q$  such that  $2 \log q$  bits of information give us  $n \log q$  bits of  $x$

Hence the effective dimension of a Liouville number is 0  
[Staiger]

This line of reasoning can be generalized to obtain

$$\dim_{\text{H}}(x) \leq \frac{2}{\delta(x)} \quad [\text{Calude \& Staiger}].$$

# Diophantine complexity

We can reformulate the irrationality exponent as an “effective dimension”.

For  $x \in \mathbb{R}$ , let

$$K_n(x) = \min\{K(p/q) : |x - p/q| \leq 2^{-n}\}.$$

(the Kolmogorov complexity at level  $n$ ).

Then

$$\dim_{\text{H}}(x) = \liminf_n \frac{K_n(x)}{n}$$

[Lutz & Mayordomo]

Similarly, let

$$D_n(x) = \min\{2 \log q : \exists p |x - p/q| \leq 2^{-n}\}.$$

(the **Diophantine** complexity at level  $n$ ).

Then

$$\delta(x) = \liminf_n \frac{D_n(x)}{n}$$

# Kolmogorov approximation

This analogy works both ways.

Let

$$\beta(x) = \sup\{b : \exists^\infty e \ |\varphi_e(0) - x| < 1/e^b\}.$$

Then  $\dim_{\text{H}} x = 1/\beta(x)$ .

[Slaman]

**Question:** How “strong” is diophantine complexity? Are Hausdorff/effective dimension and irrationality exponent completely independent?

Can reals have effective dimension  $\beta$  for any  $0 \leq \beta \leq 2/\delta(x)$ ?



## Theorem 1

Let  $\delta \geq 2$ . For every  $\beta \in [0, 2/\delta]$  there is a Cantor-like set  $E$  such that  $\dim_H(E) = \beta$  and for the uniform measure on  $E$ , almost all real numbers have irrationality exponent  $\delta$ .

## Features Of The Proof

Basic approach is to "thin" Jarník's fractal – use less intervals at each step.

However, in a straightforward way this only gets us down to dimension  $1/\delta$ .

We can get past this barrier by choosing only a uniformly spaced subset of  $G_q(\delta)$  for a single  $q$  each step.

Another problem is that the thinning might concentrate the measure no longer on reals of irrationality exponent  $\delta$ .

# Construction Template

Define a family of Cantor sets

$$\mathcal{E}(\vec{q}, \vec{m}, \vec{\delta})$$

- $\vec{q}$ : controls the choice of subintervals (thinning)
- $\vec{m}$ : controls the branching ratio
- $\vec{\delta}$ : controls the irrationality exponent (width of the intervals)

We show that for each  $\beta, \delta, \beta \leq 2/\delta$ , one can find suitable  $\vec{q}, \vec{m}, \vec{\delta}$  such that every fractal in  $\mathcal{E}(\vec{q}, \vec{m}, \vec{\delta})$  is a subfractal of the corresponding Jarník fractal  $\mathcal{E}(\vec{m}, \vec{\delta})$  and has Hausdorff dimension  $\beta$ .

# Construction Template

The family  $\mathcal{E}(\vec{q}, \vec{m}, \vec{\delta})$  can be seen as a **tree of Cantor sets**, since identical initial thinning choices up to stage  $n$  will lead to identical fractals at stage  $n$ .

Use a measure-theoretic pigeonhole argument to construct a **path through  $\mathcal{E}(\vec{q}, \vec{m}, \vec{\delta})$**  so that the resulting Cantor set has negligible measure on reals with irrationality exponent  $> \delta$ .

## Theorem 2

Let  $\delta \geq 2$ . For every  $\beta \in [0, 2/\delta]$  there is a Cantor-like set  $E$  such that for the uniform measure on  $E$ , almost all real numbers have irrationality exponent  $\delta$  and effective dimension  $\beta$ .

# Construction Template

Modifications needed:

- Since  $\delta$  and  $\beta$  are arbitrary real numbers, we have to work with approximations rather than the numbers directly.
- Ensure that the compressibility ratio of every member of  $E$  obeys the appropriate upper bound.
- To this end, exhibit a uniformly computable map taking binary sequences of a fixed, computable length onto the  $k$ -th step in the construction of  $E$ .

# Phase Transition?

How **gradual** is this independence?

If we allow more powerful approximation, will we see it vanish?

**Baker and Schmidt** extended Jarník's result to approximation by algebraic numbers.

It turns out the stratification with respect to Hausdorff dimension persists, but an analogue to the Jarník fractal is much harder to exhibit (**regular systems**).

Diophantine approximation in the **Grzegorzczuk hierarchy**?

## Randomness vs Structure

While random reals provide a vast class of numbers that cannot be well approximated, another instance is provided by **algebraic irrationals**.

Let  $\alpha$  be algebraic of degree  $d$ .

- **Liouville**:  $\delta(\alpha) \leq d$  [1844]
- **Thue**:  $\delta(\alpha) \leq \frac{1}{2}d + 1$  [1909]
- **Siegel**:  $\delta(\alpha) \leq 2\sqrt{d}$  [1921]
- **Roth**:  $\delta(\alpha) = 2$  [1955]



Thue's method is ineffective.

Given a solution  $p/q$  with sufficiently large  $q$ , he shows that the existence of another solution  $r/s$  with  $s > q$  will lead to a contradiction.

But the solution  $p/q$  may not exist at all.

In particular, for  $\delta > 1 + d/2$ , the exact, finite number of solutions cannot be extracted from the proof.

**Davenport:** There exist a primitive recursive functions  $\kappa(d)$  ( $\frac{1}{2}d + 1 < \kappa(d) < d$ ) and  $q(x, y)$  such that if  $\kappa > \kappa(d)$  and  $\alpha$  is algebraic of degree  $d$ ,

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^\kappa}$$

has **at most** one solution  $q > q(\alpha, \kappa)$ .

Upper bounds of this kind have subsequently been obtained for Roth's theorem, too, as well as explicit bounds on the number of solutions [e.g. Bombieri & Davenport, Luckhardt, Bombieri & van der Poorten, Silverman]

## Question

Is the function

$$(\alpha, \varepsilon) \mapsto \# \text{ of solutions to } \left| \alpha - \frac{p}{q} \right| < \frac{1}{q^{2+\varepsilon}}$$

computable?

# Generalized Complexity Measures

Let  $(M, \cdot, 1)$  be a monoid.

We call a function  $C : M \rightarrow \mathbb{R}^{\geq 0} \cup \{\infty\}$  a **complexity measure** if the following hold:

- $C(1) = 0$
- $C(xy) \leq C(x) + C(y)$  for all  $x, y \in M$
- $C(x) \leq C(xy) + C(y)$  for all  $x, y \in M$
- $C(xy) = C(yx)$  for all  $x, y \in M$

The pair  $(M, C)$  is called a **C-monoid**.

## Examples

- Prefix-free Kolmogorov complexity (up to an additive constant) on set of strings with concatenation.
- For a Zariski-closed subset  $X$  of  $k^n$ , let  $C(X)$  be the least degree of a polynomial vanishing identically on  $X$ .
- Any monoid homomorphism  $C : M \rightarrow \mathbb{R}^{\geq 0} \cup \{\infty\}$  is a complexity measure on  $M$ . For example, the degree map on the multiplicative monoid  $k[x_1, \dots, x_n]$ .
- Finite measurable partitions of  $[0, 1]$  under refinement with entropy.

## Dimension vs Irrationality Exponent

Suppose  $M$  is dense in a metric space  $(X, d)$  and for each  $r > 0$ , the set  $M_r = \{m \in M : C(m) \leq r\}$  is finite. For  $x \in X$ , let

$$C_n(x) = \inf\{C(m) : d(x, m) \leq 2^{-n}\}$$

and

$$\bar{C}(x) = \sup\{\beta : \exists^\infty m d(x, m) \leq 2^{-\beta C(m)}\}.$$

### Theorem

For all  $x \in X \setminus M$ ,

$$\liminf_n \frac{C_n(x)}{n} = \bar{C}(x)^{-1}.$$

# Generalized Liouville Theorem

## Theorem

Let  $(M, C)$  be a  $c$ -monoid, let  $(X, d)$  be a metric space containing  $M$ . Suppose that  $x \in X$  is such that there is some locally Lipschitz, continuous function  $f$  on  $X$  having  $x$  as an isolated zero such that there is some  $e > 0$  with  $|f(m)| \geq C(m)^{-e}$  for all  $m \in M$  in a neighborhood of  $x$ . Then there is a constant  $D > 0$  such that for all  $m \in M$  in a neighborhood of  $x$ ,  $d(x, m) \geq DC(m)^{-e}$ .

(If  $\inf\{C(m) : m \in M, m \neq 0\} > 0$ , then this holds for all  $m \in M$ .)

Thank You